Resource Sharing Models –
A mathematical description

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1. Introduction

This paper shines the spotlight on the mathematical formulae of resource sharing models. It contributes to greater transparency and comparability through a uniform mathematical representation, by showing generalisations and mergers as well as similarities and differences between currently used models. It also contains mathematical proofs for specified properties of the models.

In Chapter 2 we consider models with a limited convergence period, at the end of which global emissions are allocated to countries according to population only. The Smooth Pathway Approaches in Chapter 3 calculates national pathways starting from allocated remaining national budgets. The Emission Probability Model in Chapter 4 determines country specific emission density functions and caps the emissions of individuals.

2. Convergence models

All convergence models presented here start with a global pathway that meets a remaining global budget usually corresponding to a certain degree of global warming. Then the models break down the annual global emissions on country level, transforming the actual emissions in a base year (*BY*) into emissions based on a per capita allocation in a convergence year (*CY*) at the end of a limited convergence period

2.1 Models breaking down the global pathway in a simple way

2.1.1 Contraction & Convergence Model

The Global Commons Institute already propounded the following Contraction & Convergence Model (C&C Model) in the early 1990s. This model defines the emissions of country *i* in the year *t* $\left(\widehat{E_t^i}\right)$ recursively (cf. Meyer, No date):

$$\widehat{E_t^i} := \begin{cases} \left((1-\widehat{C}_t) * \frac{\widehat{E_{t-1}^i}}{E_{t-1}} + \widehat{C}_t * \frac{P_t^i}{P_t}\right) * E_t, \text{for } BY+1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t, \text{ for } CY \leq t \end{cases}, \quad (1)$$

where

E_t global emissions in the year *t*,

P_t global population in the year *t* and

P_t^i population of country *i* in the year *t*.

$\widehat{C}_t$ denotes the weight of the population when allocating global emissions to countries.

The Global Commons Institute considered two specifications of $\widehat{C}_t$:

- exponential (C&C-exp): $\widehat{\mathrm{C}}_\mathrm{t} = \exp\left(-\mathrm{a}\left(1-\frac{\mathrm{t-BY}}{\mathrm{CY-BY}}\right)\right)$ with the parameter a > 0 to be determined. “*The higher the value [a], the more the convergence happens towards the end of the convergence period, and vice-versa. Choosing a = 4 gives an even balance.*” (Meyer, 1998, p. 21)
- linear (C&C-lin): $\widehat{C}_t = \frac{t-BY}{CY-BY}$.

2.1.2 LIMITS Model

LIMITS, a research project funded by the EU, uses the following formula for the emissions of country i in the year t $\left(\widetilde{E_t^i}\right)$ (cf. Tavoni, et al., 2013):

$$\widetilde{E_t^i} := \begin{cases} \left(\left(1-\widetilde{C_t}\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_t} * \frac{P_t^i}{P_t}\right) * E_t, & \text{for } BY + 1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t, & \text{for } CY \leq t \end{cases} \tag{2}$$

$\widetilde{C_t}$ denotes the weight of the population when allocating global emissions to countries. LIMITS considered only the linear specification of $\widetilde{C_t}$ ($\widetilde{C_t} = \frac{t-BY}{CY-BY}$).

The LIMITS Model (LIMITS) uses formula (2) to determine emissions pathways for different regions of the world.

2.1.3 Generalised C&C Model and Generalised LIMITS Model

C&C and LIMITS consider only certain specifications of C_t. However, any non-decreasing weighting function C_t that takes the value 1 in the convergence year (*CY*) can be used. Numerous such weighting functions are conceivable. Thus we obtain the Generalised Contraction & Convergence Model (G-C&C) and the Generalised LIMITS Model (G-LIMITS). National emissions pathways with weighting functions that take the value 0 (or approximately 0) in the base year (*BY*) normally do not have a step after the base year. Therefore we only list the most intuitive weighting functions with this property:

- linear (lin): $C_t = \frac{t-BY}{CY-BY}$ (C&C-lin and LIMITS)
- exponential (exp_a): $C_t = exp\left(-a\left(1-\frac{t-BY}{CY-BY}\right)\right)$ with the parameter $a > 0$ to be determined (C&C-exp)
- convex quadratic (conv quadr): $C_t = \left(\frac{t-BY}{CY-BY}\right)^2$
- concave quadratic (conc quadr): $C_t = 1-\left(1-\frac{t-BY}{CY-BY}\right)^2$
- general quadratic: $C_t = a(t-BY)^2 + b(t-BY) + c$, where a, b and c are parameters to be determined in such a way that $C_{BY} = 0$, $C_{CY} = 1$ and with a third constraint, e. g. a given value for the year after the base year. The linear, the convex quadratic and the concave quadratic specifications of C_t are special cases of the general quadratic specification.
- cubic: $C_t = -2\left(\frac{t-BY}{CY-BY}\right)^3 + 3\left(\frac{t-BY}{CY-BY}\right)^2$

- convex polynomial (conv pol_n): $C_t = \left(\frac{t - BY}{CY - BY}\right)^n$, where *n* is a natural number

- concave polynomial (conc pol_n): $C_t = 1 - \left(1 - \frac{t - BY}{CY - BY}\right)^n$, where *n* is a natural number

The weighting functions above depend directly on the year (*t*). Another class of weighting functions is obtained by introducing the emissions in the year *t* (E_t). Thus these weighting functions depend on the global emissions and only indirectly on the year. We only show the linear specification as an example: linear in E_t (lin_E_t): $C_t = \frac{E_{BY} - E_t}{E_{BY} - E_{CY}}$.

Figure 1 depicts the trajectories of some weighting functions.

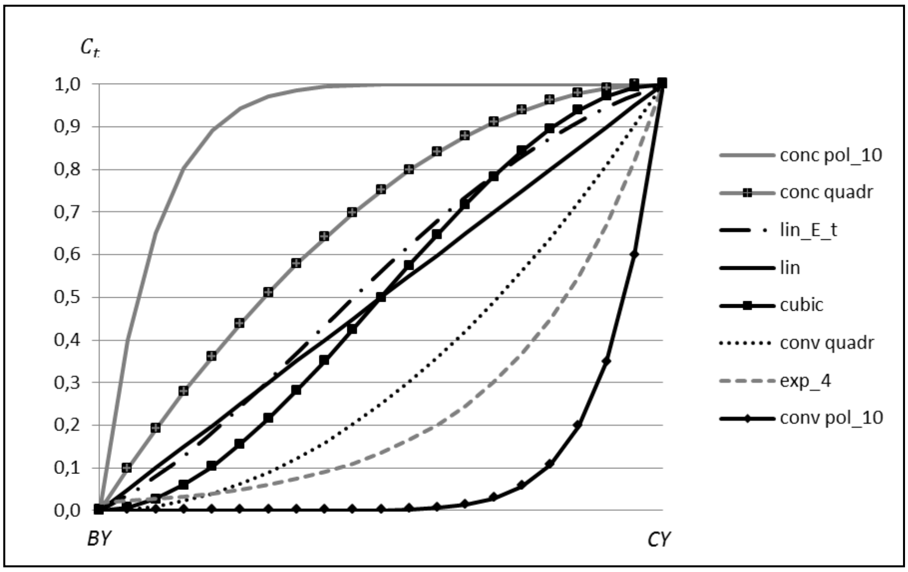

Figure 1: Trajectories of the different specifications of C_t

Figure 1 shows that, if *n* is great enough, the allocation key "population"

- in the concave polynomial specification comes fully into effect already in the first year after the base year (equity, immediate climate justice).

- in the convex polynomial specification comes into effect only in the convergence year (inertia).

2.1.4 Common but Differentiated Convergence Model

The Common but Differentiated Convergence Model is described in (cf. Höhne, et al., 2006). This source does not contain any formulae, so the formulae presented here are our interpretation of the description of the CDC model.

First a threshold TH_t in the year *t* is defined, which decreases if the global emissions decrease:

$TH_t := \frac{E_t}{P_t} * PT,$

where PT is a given percentage, e. g. 0.95. If the average emissions of country *i* in the year *t* in a business as usual scenario $\left(\frac{E_t^{i\_bau}}{P_t^i}\right)$ are below or equal to the threshold, i. e. $\frac{E_t^{i\_bau}}{P_t^i} \leq TH_t$, the country is allocated emissions according to the business as usual scenario and we define

$E_t^i := E_t^{i\_bau}.$

Otherwise, if the average emissions of country *i* in the year *t* in the business as usual scenario are above the threshold ($\frac{E_t^{i\_bau}}{P_t^i} > TH_t$), the country is allocated emissions according to the C&C formula and we define

$E_t^i := \left((1 - \widehat{C}_t) * \frac{E_{t-1}^i}{E_{t-1}^{oTH\_t}} + \widehat{C}_t * \frac{P_t^i}{P_t^{oTH}}\right) * E_t^{oTH},$

where

$\widehat{C}_t$ weighting of per capita emissions in the year *t*,

E_t^{oTH} remaining emissions in the year *t* for the countries over the threshold in the year *t*, i. e.

$$E_t^{oTH} = E_t - \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} \leq TH_t}} E_t^i \, ,$$

$E_{t-1}^{oTH\_t}$ emissions in the year *t*-1 of the countries over the threshold in the year *t*, i. e.

$$E_{t-1}^{oTH\_t} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} E_{t-1}^i \text{ and}$$

P_t^{oTH} population in the year *t* of the countries over the threshold in the year *t*, i. e.

$$P_t^{oTH} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} P_t^i.$$

Remark: Obviously the equation

$$E_t^{oTH} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} E_t^i$$

holds, but this equation cannot be used to define E_t^{oTH}, because E_t^i is defined with the help of E_t^{oTH}.

2.2 The Regensburg Formula (RF)

We will present three equivalent notations of the Regensburg Formula

- as a weighting function with an annual degree of achieving the global convergence amount
- as a straight line with a conversion factor for the reduction of emissions
- as a recursion with an annual rate of change

and show how they are derived from each other. Then we show the derivation of a formula for the national budget in the convergence period of an individual country.

2.2.1 The RF as a weighting function

The notation of the **RF as a weighting function** (cf. Sargl, et al., 2017) uses the annual degree of achieving the global convergence amount E_{CY} in year *t*

$$\overline{\boldsymbol{C_t}} := \frac{\boldsymbol{E_{BY} - E_t}}{\boldsymbol{E_{BY} - E_{CY}}}$$

as weighting factor for the national convergence amount E_{CY}^i (in case of the national convergence amount being directly proportional to the population, it is also a per-capita weighting factor) for the calculation of emissions of country *i* in year *t*:

$$\overline{\boldsymbol{E_t^i}} := (\boldsymbol{1} - \overline{\boldsymbol{C_t}}) * \boldsymbol{E_{BY}^i} + \overline{\boldsymbol{C_t}} * \boldsymbol{E_{CY}^i}, \quad BY + 1 \leq t \leq CY$$

Directly from this definition of the RF we obtain the following results:

Remark 1 (equal proportions in all countries and the world)

In each year *t*, the proportion of emissions still to be reduced and the proportion of emissions already reduced in relation to the emissions to be reduced altogether are equal in all countries and globally:

$$\frac{E_t - E_{CY}}{E_{BY} - E_{CY}} = \frac{\overline{E_t^i} - E_{CY}^i}{E_{BY}^i - E_{CY}^i} \; (= 1 - \overline{C_t}) \text{ and}$$

$$\frac{E_{BY} - E_t}{E_{BY} - E_{CY}} = \frac{E_{BY}^i - \overline{E_t^i}}{E_{BY}^i - E_{CY}^i} \; (= \overline{C_t}).$$

In each year *t*, therefore, the degree of achieving the global convergence amount and the degree of achieving the national convergence amount are identical.

Remark 2 (national convergence amounts in all countries in CY)

In *CY* emissions calculated with the RF and the national convergence amount are the same in each country.

Remark 3 (Uniqueness of $\overline{C_t}$)

There is only one weighting function $\overline{C_t}$ so that the equation

$$\boldsymbol{E_t^i = (1 - \overline{C_t}) * E_{BY}^i + \overline{C_t} * E_{CY}^i}$$

holds for each country. This weighting function is $\overline{C_t} := \frac{E_{BY} - E_t}{E_{BY} - E_{CY}}$. This can be shown by summing up the equation across all countries, yielding an equation that can be solved for $\overline{C_t}$.

2.2.2 The RF as a straight line

Theorem 1 (notation of the RF as a straight line)

The emissions of each country *i* as a function of the global emissions are on a straight line:

$$\boldsymbol{\overline{E_t^i} = (E_t - E_{CY}) * a^i + E_{CY}^i}, \quad BY + 1 \le t \le CY,$$

with the conversion factor for the reduction: $\boldsymbol{a^i} := \frac{\boldsymbol{E_{BY}^i - E_{CY}^i}}{\boldsymbol{E_{BY} - E_{CY}}}$.

Proof:

$$E_t^i =$$

$$= E_{BY}^i * (1 - \overline{C_t}) + \overline{C_t} * E_{CY}^i =$$

$$= E_{BY}^i * \left(1 - \frac{E_{BY} - E_t}{E_{BY} - E_{CY}}\right) + \left(\frac{E_{BY} - E_t}{E_{BY} - E_{CY}}\right) * E_{CY}^i =$$

$$= E_{BY}^i * \left(\frac{E_t - E_{CY}}{E_{BY} - E_{CY}}\right) + \left(1 - \frac{E_t - E_{CY}}{E_{BY} - E_{CY}}\right) * E_{CY}^i =$$

$$= (E_t - E_{CY}) * \frac{E^i_{BY} - E^i_{CY}}{E_{BY} - E_{CY}} + E^i_{CY} =$$

$$= (E_t - E_{CY}) * a^i + E^i_{CY}$$

□

Remark 3 (stepwise approximation)

By presenting the RF as a straight line, it becomes clear that a stepwise approximation of the global emission pathway to the global convergence amount is transmitted to all national emission pathways.

Remark 4 (construction of national graphs)

This theorem also shows that, when applying the RF, the national graph (t, $\overline{E^i_t}$) for country *i* with a reduction amount ($E^i_{BY} > E^i_{CY}$) can be derived from the global graph (t, E_t) by changing the scaling on the ordinate and by vertically shifting the abscissa. For countries with a national convergence amount permitting increasing annual emissions ($E^i_{BY} < E^i_{CY}$), the global graph additionally needs to be reflected across the abscissa to obtain the national graph.

Remark 5 (factor for converting reductions = proportional factor)

Because of $\sum_i a^i = 1$ the factor for converting the reduction is also called “proportional factor”.

Corollary 1 (constant factor for converting reductions)

For each country *i* there is a constant proportional factor α^i that allows converting annual global reductions to annual reductions of country *i*:

$$\overline{E^i_t} - \overline{E^i_{t-1}} = (E_t - E_{t-1}) \ * \ a^i.$$

Factor a^i for converting reductions can be determined by the ratio between emissions that remain to be reduced by country *i* in year *t* and emissions which remain to be reduced globally:

$$a^i = \frac{\overline{E^i_t} - E^i_{CY}}{E_t - E_{CY}} \ \ (BY \leq t \leq CY - 1).$$

Remark 6 (monotonicity)

This corollary also shows that monotonicity of the global emission pathway is transferred to the national emission pathways.

Corollary 2 (complete distribution of global emissions)

The emissions determined according to the RF of all countries together sum up to the amount of global emissions:

$$\sum_i \overline{E_t^i} = E_t \quad \text{for every year } t.$$

Proof by using the notation of the RF as a straight line:

$$\sum_i \overline{E_t^i} =$$

$$= \sum_i \left((E_t - E_{CY}) * a^i + E_{CY}^i \right) =$$

$$= (E_t - E_{CY}) * \sum_i a^i + \sum_i E_{CY}^i =$$

$$= (E_t - E_{CY}) * 1 + E_{CY} = E_t$$

□

2.2.3 *The RF as a recursion*

Theorem 2 (notation of the RF as a recursion)

We have:[1]

$$\boldsymbol{\overline{E_t^i} = \overline{E_{t-1}^i} - CR_{t-1} * (\overline{E_{t-1}^i} - E_{CY}^i)}, \qquad BY + 1 \leq t \leq CY$$

$$\text{with the annual rate of change } \boldsymbol{CR_{t-1}} := \frac{\boldsymbol{E_{t-1} - E_t}}{\boldsymbol{E_{t-1} - E_{CY}}}.$$

Proof:

CR_{t-1} is well defined, because $E_{t-1} \neq E_{CY}$ for $BY + 1 \leq t \leq CY$.

By using corollary 1 for the factor for converting reductions, we can say:

$$\overline{E_t^i} =$$

$$= \overline{E_{t-1}^i} + (E_t - E_{t-1}) * a^i =$$

$$= \overline{E_{t-1}^i} - \frac{E_{t-1} - E_t}{E_{t-1} - E_{CY}} * \left(\overline{E_{t-1}^i} - E_{CY}^i\right) =$$

$$\overline{= E_{t-1}^i} - CR_{t-1} * (\overline{E_{t-1}^i} - E_{CY}^i)$$

□

[1] Alternative notation with $TA := E_{CY}, TA^i := E_{CY}^i$ and $\widetilde{CR}_{t-1} := -CR_{t-t} = \frac{E_t - E_{t-1}}{E_{t-1} - TA}$: $\overline{E_t^i} = \overline{E_{t-1}^i} + \widetilde{CR}_{t-1} * (\overline{E_{t-1}^i} - TA^i)$.

Remark 7 (identical annual rates of change)

The notation as a recursion offers another interpretation of the RF: The annual emissions of country *i* in the year *t* are determined by transferring the rates of change which are derived from the global emission pathway, to national emission pathways. Therefore, in each year t, the national and global annual rates of change are identical.

Remark 8 (national convergence amounts in all countries in the convergence year)

From the notation of the RF as a recursion, you can see that the convergence amounts are achieved in all countries in the year CY, if you take into consideration that the rate of change CR_{CY-1} takes value 1.

2.2.4 Implicit national budgets

The emissions of country *i* until the year *t* are denominated national budget of country *i* until the year *t*:

$$B_t^i := \sum_{l=BY+1}^{t} E_l^i.$$

The global emissions until the year *t* are denominated global budget until the year *t*:

$$B_t := \sum_{l=BY+1}^{t} E_l \left(= \sum_{l=BY+1}^{t} \sum_i E_l^i = \sum_i \sum_{l=BY+1}^{t} E_l^i = \sum_i B_t^i \right).$$

Theorem 3 (national budget in the convergence period)

For the national budget of country *i* in the convergence period we have:

$$B^i = E_{CY}^i * (CY - BY) + (B - E_{CY} * (CY - BY)) * a^i,$$

$$\text{with the factor } a^i = \frac{E_{BY}^i - E_{CY}^i}{E_{BY} - E_{CY}} \text{ for converting reductions.}$$

Proof:

According to the notation of the RF as a straight line, the following applies to the emissions of country *i* in year *t*:

$$\overline{E_t^i} = (E_t - E_{CY}) * a^i + E_{CY}^i.$$

By summing up these emissions across all years, we obtain the national budget of country *i* in the convergence period:

$$B^i = \sum_{t=BY+1}^{CY} \overline{E_t^i} =$$

$$= \sum_{t=BY+1}^{CY} E_{CY}^i + \sum_{t=BY+1}^{CY} (E_t - E_{CY}) * a^i$$

$$= E_{CY}^i * (CY - BY) + (B - E_{CY} * (CY - BY)) * a^i$$

□

Remark 9 (national budget depending only on the global budget)

This theorem also shows, that the national budget of country *i* in the convergence period only depends on – besides the national emissions of country *i* and the global emissions in *BY* and in *CY* – the global budget in the convergence period, but not on the global emissions $E_{BY+1}, E_{BY+2}, \dots, E_{CY-2}, E_{CY-1}$. Under certain conditions, this results in an implicit national budget that is independent of the global path chosen.

2.3 Convertibility of the convergence models

2.3.1 Equivalence of the Generalised C&C and LIMITS Model

In both models, the population is frozen and the convergence amount of a country *i* is defined by $E_{CY}^i = \frac{P^i}{P} * E_{CY}$.

G-C&C is given by

$$\widehat{E_t^i} := \left((1 - \widehat{C_t}) * \frac{\widehat{E_{t-1}^i}}{E_{t-1}} + \widehat{C_t} * \frac{P^i}{P} \right) * E_t, \text{for } BY + 1 \leq t \leq CY$$

with a weighting function $\widehat{C_t}$ that takes the value 0 (or approximately 0) in *BY* and the value 1 in the *CY*. Here $\widehat{E_t^i}$ is defined recursively.

The G-Limits is given by

$$\widetilde{E_t^i} := \left(\left(1 - \widetilde{C_t}\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_t} * \frac{P^i}{P} \right) * E_t, \text{for } BY + 1 \leq t \leq CY$$

with a weighting function $\widetilde{C_t}$ that takes the value 0 (or approximately 0) in *BY* and the value 1 in *CY*.

Theorem 4 (equivalence of G-C&C and G-LIMITS)

For any weighting function $\widehat{C_t}$ of G-C&C there is a weighting function $\widetilde{C_t}$ for G-LIMITS, so that the results of G-C&C and G-LIMITS are the same.

For any weighting function $\widetilde{C_t}$ of G-LIMITs there is a weighting function $\widehat{C_t}$ for G-C&C, so that the results of G-C&C and G-LIMITS are the same.

Proof:

If we know the weighting function $\widehat{C_t}$ of G-C&C, the weighting function $\widetilde{C_t}$ of G-LIMITS is given by

$$\widetilde{C_t} \coloneqq 1 - \prod_{l=BY+1}^{t} \left(1 - \widehat{C_l}\right) \text{ for } BY + 1 \leq t \leq CY\,.$$

We proof the first part of the theorem by aid of mathematical induction.

Base case: For $t = BY + 1$ we obtain $\widetilde{C_{BY+1}} = \widehat{C_{BY+1}}$ and

$$\widetilde{E_{BY+1}^i} := \left(\left(1 - \widetilde{C_{BY+1}}\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_{BY+1}} * \frac{P^i}{P}\right) * E_{BY+1}$$

$$= \left((1 - \widehat{C_{BY+1}}) * \frac{E_{BY}^i}{E_{BY}} + \widehat{C_{BY+1}} * \frac{P^i}{P}\right) * E_{BY+1} = \widehat{E_{BY+1}^i}$$

Inductive step: Assuming that if $\widehat{E_{t-1}^i} = \widetilde{E_{t-1}^i} = \left(\left(1 - \widetilde{C_{t-1}}\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^i}{P}\right) * E_{t-1}$, we show that $\widehat{E_t^i} = \widetilde{E_t^i}$. Algebraically

$$\widehat{E_t^i} = \left((1 - \widehat{C_t}) * \frac{\widehat{E_{t-1}^i}}{E_{t-1}} + \widehat{C_t} * \frac{P^i}{P}\right) * E_t,$$

$$= \left((1 - \widehat{C_t}) * \frac{\left(\left(1 - \widetilde{C_{t-1}}\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^i}{P}\right) * E_{t-1}}{E_{t-1}} + \widehat{C_t} * \frac{P^i}{P}\right) * E_t$$

$$= \left((1 - \widehat{C_t}) * \left(\left(1 - \widetilde{C_{t-1}}\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^i}{P}\right) + \widehat{C_t} * \frac{P^i}{P}\right) * E_t$$

$$= \left((1 - \widehat{C_t}) * \left(\left(1 - \left(1 - \prod_{l=BY+1}^{t-1} \left(1 - \widehat{C_l}\right)\right)\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^i}{P}\right) + \widehat{C_t} * \frac{P^i}{P}\right) * E_t$$

$$= \left((1-\widehat{C_t}) * \left(\left(\left(\prod_{l=BY+1}^{t-1} \left(1-\widehat{C_l}\right) \right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_{t-1}} * \frac{P^i}{P} \right) + \widehat{C_t} * \frac{P^i}{P} \right) * E_t$$

$$= \left(\prod_{l=BY+1}^{t} \left(1-\widehat{C_l}\right) * \frac{E_{BY}^i}{E_{BY}} + (1-\widehat{C_t}) * \left(1 - \prod_{l=BY+1}^{t-1} \left(1-\widehat{C_l}\right) \right) * \frac{P^i}{P} + \widehat{C_t} * \frac{P^i}{P} \right) * E_t$$

$$= \left(\left(1-\widetilde{C_t}\right) * \frac{E_{BY}^i}{E_{BY}} + \left((1-\widehat{C_t}) - \left(1-\widetilde{C_t}\right) \right) * \frac{P^i}{P} + \widehat{C_t} * \frac{P^i}{P} \right) * E_t$$

$$= \left(\left(1-\widetilde{C_t}\right) * \frac{E_{BY}^i}{E_{BY}} + \widetilde{C_t} * \frac{P^i}{P} \right) * E_t = \widetilde{E_t^i}$$

Second part of the theorem: If we know the weighting function $\widetilde{C_t}$ of G-LIMITS, we solve the definition of $\widetilde{C_t}$ for $\widehat{C_t}$ and obtain recursively the weighting function $\widehat{C_t}$ of G-C&C:

$$\widehat{C_t} = 1 - \frac{1-\widetilde{C_t}}{\prod_{l=BY+1}^{t-1}\left(1-\widehat{C_l}\right)} \text{ for } BY + 1 \le t \le CY$$

$\widehat{C_t}$ is well defined because *CY* is by definition the year when the convergence amount is reached.

□

2.3.2 *RF as a special case of the Generalised C&C and LIMITS Model*

Theorem 5 (The RF as a special case of G-LIMITS)

With the weighting function

$$\widetilde{C_t} = \frac{\frac{\overline{E_t^i}}{E_t} - \frac{E_{BY}^i}{E_{BY}}}{\frac{E_{CY}^i}{E_{CY}} - \frac{E_{BY}^i}{E_{BY}}}$$

the results of G-LIMITS and the RF are the same.

Proof:

The weighting function $\widetilde{C_t}$ is obtained by transforming G-LIMITS for country *i* using $\frac{P^i}{P} = \frac{E_{CY}^i}{E_{CY}}$ and assuming that $\widetilde{E_t^i} = \overline{E_t^i}$. Thus we have to proof that we obtain the same weighting function $\widetilde{C_t}$ for any other country *j:*

$$\frac{\frac{\overline{E_t^i}}{E_t} - \frac{\overline{E_{BY}^i}}{E_{BY}}}{\frac{E_{CY}^i}{E_{CY}} - \frac{E_{BY}^i}{E_{BY}}} = \frac{\frac{\overline{E_t^j}}{E_t} - \frac{\overline{E_{BY}^j}}{E_{BY}}}{\frac{E_{CY}^j}{E_{CY}} - \frac{E_{BY}^j}{E_{BY}}}$$

$$\frac{\overline{E_t^i} * E_{BY} - E_{BY}^i * E_t}{E_t * E_{BY}} * \frac{E_{CY} * E_{BY}}{E_{CY}^i * E_{BY} - E_{BY}^i * E_{CY}} = \frac{\overline{E_t^j} * E_{BY} - E_{BY}^j * E_t}{E_t * E_{BY}} * \frac{E_{CY} * E_{BY}}{E_{CY}^j * E_{BY} - E_{BY}^j * E_{CY}}$$

$$0 = \left(\overline{E_t^i} * E_{BY} - E_{BY}^i * E_t\right) * \left(E_{CY}^j * E_{BY} - E_{BY}^j * E_{CY}\right)$$
$$-\left(\overline{E_t^j} * E_{BY} - E_{BY}^j * E_t\right) * \left(E_{CY}^i * E_{BY} - E_{BY}^i * E_{CY}\right).$$

Since $\overline{E_t^i}$ and $\overline{E_t^j}$ can be seen as a function of E_t whose images are on a straight line (Theorem 1), the right side of this equation can be seen as a function of E_t whose image is on a straight line. Therefore, it is sufficient to proof that two points of the image are 0. These two points are obviously E_{BY} and E_{CY}.

□

Remark 10 (The RF as a special case of G-C&C)

Since the results of G-LIMITS can be obtained with G-C&C using an appropriate weighting function (theorem 4), The RF is also a special case of G-C&C.

2.4 Implicit weighting of the population in convergence models

Each convergence model allocates a country *i* until the year *t* a national budget that can be considered as a weighting of the two extreme allocations “emissions in the past” and “frozen population”:

$$B_t^i = \left(\left(1 - \check{C}_t^i\right) * \frac{E_{BY}^i}{E_{BY}} + \check{C}_t^i * \frac{P^i}{P}\right) * B_t \tag{3}$$

Theorem 6 (Identical weighting of the population in all countries)

If the population is frozen, then for any convergence model, the population weighting is the same for each country: $\check{C}_t^i = \check{C}_t$. Convergence models thus show an implicit weighting of the population.[2]

[2] In our tool for the Regensburg Model, this implicit weighting can be calculated for different framework data.

Proof:

We proof this theorem for the G-LIMITS by aid of mathematical induction. The rest follows from the equivalence of G-C&C and G-LIMITS (theorem 4) and the fact that the RF is a special case of G-LIMITS (theorem 5).

Base case: For $t = BY + 1$ the national budget of country i until the year $BY + 1$ is E^i_{BY+1} and the global budget is E_{BY+1}. By comparing equation (2) with equation (3) we obtain $\check{C}^i_{BY+1} = \widetilde{C_{BY+1}}$.

Inductive step: Assuming that if $\check{C}^i_{t-1} = \check{C}_{t-1}$ for each country i we show that $\check{C}^i_t = \check{C}_t$.

For the national budget of each country i until the year t we obtain

$$B^i_t = E^i_t + B^i_{t-1} = E^i_t + \left(\left(1 - \check{C}^i_{t-1}\right) * \frac{E^i_{BY}}{E_{BY}} + \check{C}^i_{t-1} * \frac{P^i}{P}\right) * B_{t-1} =$$

$$= \left(\left(1 - \widetilde{C}_t\right) * \frac{E^i_{BY}}{E_{BY}} + \widetilde{C}_t * \frac{P^i}{P}\right) * E_t + \left(\left(1 - \check{C}_{t-1}\right) * \frac{E^i_{BY}}{E_{BY}} + \check{C}_{t-1} * \frac{P^i}{P}\right) * B_{t-1} =$$

$$= \left(E_t + B_{t-1} - \widetilde{C}_t * E_t + \check{C}_{t-1} * B_{t-1}\right) * \frac{E^i_{BY}}{E_{BY}} + \left(\widetilde{C}_t * E_t + \check{C}_{t-1} * B_{t-1}\right) * \frac{P^i}{P}.$$

We define $\check{C}_t = \frac{\widetilde{C}_t * E_t + \check{C}_{t-1} * B_{t-1}}{B_t}$ and obtain

$$B^i_t = \left(B_t - \check{C}_t * B_t\right) * \frac{E^i_{BY}}{E_{BY}} + \left(\check{C}_t * B_t\right) * \frac{P^i}{P} = \left(\left(1 - \check{C}_t\right) * \frac{E^i_{BY}}{E_{BY}} + \check{C}_t * \frac{P^i}{P}\right) * B_t.$$

□

3. Smooth Pathway Models

This approach is based on a national budget derived from a global budget. For example, Raupach et al. proposed a simple weighting formula that includes emissions and the population in a base year in order to distribute a global budget across countries (cf. Raupach, et al., 2014). The following chapters describe the derivation of national paths that adhere to a given national budget. The approaches to deriving the national budgets are actually "Resource Sharing Models". The derivations of the paths are mathematically interesting.

3.1 Smooth Pathway Formula from Raupach (SPFR)

In the Smooth Pathway Formula from Raupach et al. for the emission power, i. e. the derivative of emissions with respect to time or the emissions per unit of time, of country *i* at a point of time $z \geq BJ + 1$ the following function is used (cf. Raupach, et al., 2014):

$$\dot{E}^i(z) = \dot{E}^i_{BY+1}\big(1 + (r^i + m^i)(z - BY - 1)\big)e^{-m^i(z-BY-1)}, \tag{4}$$

where

$\dot{E}^i_{BY+1}$ emission power of country *i* at the end of the base year,

r^i change rate of the emission power of country *i* at the end of the base year $\left(\frac{d\dot{E}^i}{dz}(BY+1)/\dot{E}^i(BY+1) = r^i\right)$ and

m^i the mitigation rate (or the decay parameter) of country *i*.

The mitigation rate m^i is determined such that the allocated remaining budget of country *i* (RB^i) is met:

$$\int_{BY+1}^{\infty} \dot{E}^i(z)\,dz = RB^i.$$

Thus, we obtain

$$\int_{BY+1}^{\infty} \dot{E}^i(z)\,dz =$$

$$= \int_{BY+1}^{\infty} \dot{E}^i_{BY+1}\big(1 + (r^i + m^i)(z - BY - 1)\big)e^{-m^i(z-BY-1)}\,dz =$$

$$= \dot{E}^i_{BY+1}\int_{BY+1}^{\infty} e^{-m^i(z-BY-1)}\,dz + \dot{E}^i_{BY+1}(r^i + m^i)\int_{BY+1}^{\infty} (z - BY - 1)e^{-m^i(z-BY-1)}\,dz =$$

$$= \dot{E}^i_{BY+1}\left[\frac{-1}{m^i}e^{-m^i(z-BY-1)}\right]_{z=BY+1}^{z=\infty}$$

$$+\dot{E}^i_{BY+1}(r^i+m^i)\left[\frac{-(z-BY-1)}{m^i}e^{-m^i(z-BY-1)}-\frac{1}{(m^i)^2}e^{-m^i(z-BY-1)}\right]_{z=BY+1}^{z=\infty}$$

$$= \dot{E}^i_{BY+1}\left[\frac{1}{m^i}\right]+\dot{E}^i_{BY+1}(r^i+m^i)\ \left[\frac{1}{(m^i)^2}\right]=RB^i.$$

With the time $T^i = \frac{RB^i}{\dot{E}^i_{BY+1}}$ defined by the remaining budget of country *i* and the emission power of country *i* at the end of the base year we obtain

$$T^i\left(m^i\right)^2-2m^i-r^i=0.$$

Thus, if $r^i > -1/T^i$, the mitigation rate m^i is given by

$$m^i = \frac{1+\sqrt{1+r^iT^i}}{T^i},$$

There is otherwise no solution for the mitigation rate m^i. In this rare case a simple exponential decay function is used:

$\dot{E}^i(z) = \dot{E}^i_{BY+1}e^{-m^i(z-BY-1)}$.

Since we are more interested in the emissions of country *i* in the year *t* (E^i_t) than in the emission power at a point of time *z*, we integrate equation (4) and obtain:

$$E^i_t = \int_t^{t+1}\dot{E}^i(z)\,dz =$$

$$-\dot{E}^i_{BY+1}\frac{e^{-m^i(t-BY)}}{(m^i)^2}\left[\left(r^im^i+\left(m^i\right)^2\right)(t-BY)+2m^i+r^i\right]$$

$$+\dot{E}^i_{BY+1}\frac{e^{-m^i(t-BY-1)}}{(m^i)^2}\left[\left(r^im^i+\left(m^i\right)^2\right)(t-BY-1)+2m^i+r^i\right].$$

Supplementary information containing mathematical details on the properties of the formula in equation (4) can be retrieved from http://www.nature.com/nclimate/journal/v4/n10/extref/nclimate2384-s1.pdf.

3.2 Generalised Smooth Pathway Formula (GSPF)

In order to allow for negative emission we generalise equation (4) using the following function for the emission power, i. e. the derivative of emissions with respect to time or the emissions per unit of time, of country *i* at a point of time $z \geq BJ + 1$:

$$\dot{E}^i(z) = p_\infty + \left(p_0 + p_1(z - BY - 1)\right) e^{-p_2(z-BY-1)}, \tag{5}$$

where

the parameter p_∞ is the emission power at infinity and the parameters p_0, p_1 and p_2 are determined in a way that the following constraints hold

(1) $\dot{E}^i(BY + 1) = \dot{E}^i_{BY+1}$,

(2) $\frac{d\dot{E}^i}{dz}(BY + 1)/\dot{E}^i(BY + 1) = r^i$

(3) $\int_{BY+1}^{2101} \dot{E}^i(z)\, dz = RB^i$

with

$\dot{E}^i_{BY+1}$ emission power of country *i* at the end of the base year,

r^i change rate of the emission power of country *i* at the end of the base year,

RB^i remaining budget of country *i* in the period starting at the end of the base year and ending in at the end of the year 2100.

The first constraint leads to $p_0 = \dot{E}^i_{BY+1} - p_\infty$.

The second constraint leads to $p_1 = \dot{E}^i_{BY+1} r + \left(\dot{E}^i_{BY+1} - p_\infty\right) p_2$.

The third constraint determines p_2.

The emissions of country *i* in the year *t* (E^i_t) are obtained by integrating equation (5):

$$E^i_t = \int_t^{t+1} \dot{E}^i(z)\, dz =$$

$$\left[p_\infty(z - BY - 1) - \frac{p_0}{p_2} e^{-p_2(z-BY-1)} - \frac{p_1(z-BY-1)}{p_2} e^{-p_2(z-BY-1)} - \frac{p_1}{p_2^2} e^{-p_2(z-BY-1)}\right]_{z=t}^{z=t+1}.$$

3.3 Extended Smooth Pathway Model (ESPM)

Our Extended Smooth Pathway Model[3] uses six types of scenarios to derive national paths from a national budget. These Regensburg Model Scenario Types differ in the assumption about the property of the annual changes and can also map negative emissions. For a comprehensive mathematical description of the RM Scenario Types, we refer to the corresponding paper, which can be downloaded from our website or use this direct link.

[3] Download Excel tool and papers: save-the-climate.info.

4. Emission Probability Model

Chakravarty et al. (cf. Chakravarty, et al., 2009) described three steps to obtaining and cutting an emission probability density function (PDF) starting with the points of a Lorenz curve. We hence summarize how to obtain a Lorenz Curve from a PDF in Chapter 4.1, show the results for a gamma PDF in Chapter 4.2 and describe the Emission Probability Model (EPM) in Chapter 4.3.

4.1 General case: The Lorenz Curve obtained from a PDF

Let *f* be an income PDF.

Then

- the cumulative population share *x* is given by the cumulative distribution function (CDF) *F*, i. e. the probability of an income equal to z or less is $x = F(z) = \int_{-\infty}^{z} f(t)\, dt$
- the cumulative income share *y* is given by $y = \frac{\int_{-\infty}^{z} t\, f(t)\, dt}{\int_{-\infty}^{\infty} t\, f(t)\, dt}$

  $\int_{-\infty}^{z} t\, f(t)\, dt$: average income of the persons with an income equal to *z* or less

  $\int_{-\infty}^{\infty} t\, f(t)\, dt$: average income of the population

Thus a parametric representation of the Lorenz curve $\bar{L}$ is given by

$$\bar{L}(z) = \begin{pmatrix} x = F(z) \\ y = \dfrac{\int_{-\infty}^{z} t\, f(t)\, dt}{\int_{-\infty}^{\infty} t\, f(t)\, dt} \end{pmatrix} \quad (6)$$

If the inverse function F^{-1} of the CDF *F* exists, the Lorenz curve *L* is directly given by

$$\mathrm{y} = \mathrm{L(x)} = \frac{\int_{-\infty}^{F^{-1}(\mathrm{x})} \mathrm{t\, f(t)\, dt}}{\int_{-\infty}^{\infty} \mathrm{t\, f(t)\, dt}}. \quad (7)$$

Substituting $t = F^{-1}(\check{t})$ yields $\frac{dt}{d\check{t}} = (F^{-1})'(\check{t}) = \frac{1}{F'(F^{-1}(\check{t}))} = \frac{1}{f(F^{-1}(\check{t}))}$ and the Lorenz curve can be written as

$$\mathrm{y} = \mathrm{L(x)} = \frac{\int_{0}^{\mathrm{x}} \mathrm{F}^{-1}(\check{\mathrm{t}})\mathrm{d}\check{\mathrm{t}}}{\int_{0}^{1} \mathrm{F}^{-1}(\check{\mathrm{t}})\mathrm{d}\check{\mathrm{t}}}. \quad (8)$$

Theorem 7 (Scaling)

The Lorenz curve is independent of the scaling of the z-axis.

Proof: With a scaling factor $s \neq 0$ the scaled PDF $\tilde{f}$ for a PDF *f* is given by

$$\tilde{f}(\tilde{z}) = s\, f(s\tilde{z}).$$

For the CDF $\tilde{F}$ we obtain

$$\tilde{F}(\tilde{z}) = \int_{-\infty}^{\tilde{z}} \tilde{f}(\tilde{t})\, d\tilde{t} = s\int_{-\infty}^{\tilde{z}} f(s\tilde{t})\, d\tilde{t} = \int_{-\infty}^{s\tilde{z}} f(t)dt = F(s\tilde{z}).$$

Thus $\tilde{F}^{-1}$, the inverse function of the CDF $\tilde{F}$, is given by

$$\tilde{F}^{-1} = \frac{1}{s}\, F^{-1}.$$

With the help of the representation (8). of the Lorenz curve we see that, the Lorenz curve from the PDF f and the PDF $\tilde{f}$ are the same.

4.2 Special case: The Lorenz Curve obtained from a gamma probability distribution

In general, the evaluation of the integrals in equation (1) or (2) can cause trouble. However if Z is a gamma distributed random variable all this work can be done by a spreadsheet programme, such as EXCEL.

Let Z be a gamma distributed random variable. Then the PDF g is given by

$$g(z; a, b) \;=\; \begin{cases} 0 & \text{for z } < 0 \\ \dfrac{1}{b^a\,\Gamma(a)} z^{a-1} e^{-\frac{z}{b}} & \text{for } z \geq 0 \end{cases}$$

with parameters a, $b > 0$ and $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z}\, dz$.

The CDF is denoted by

$$G(z; a, b) = \int_0^z g(t;\; a, b)\, dt = \int_0^z \frac{1}{b^a\,\Gamma(a)} t^{a-1} e^{-\frac{t}{b}}\, dt$$

Since $\Gamma(a+1) \;=\; a\;\Gamma(a)$, the equation $t\; g(t; a, b) = \; ab\; g(t; a+1, b)$ holds. Thus

- the expected value (or mean) of Z is given by

$$E[Z] \;=\; \int_0^\infty t\;\, g(t; a, b)\; dt \;=\; ab\,\int_0^\infty g(t; a+1, b)dt = ab$$

  and

- using the representation (7). the Lorenz curve is given by

$$L(x) = \frac{\int_0^{G^{-1}(x;a,b)} t\, g(t;a,b)\, dt}{\int_0^{\infty} t\, g(t,a,b)\, dt} = \frac{ab\, \int_0^{G^{-1}(x;a,b)} g(t;a+1,b)\, dt}{ab} = G(G^{-1}(x;a,b);a+1,b).$$

Scaling

With a scaling factor $s \neq 0$ we easily find

$$\tilde{g}(\tilde{z};a,b) = s\, g(s\tilde{z};a,b) = g(\tilde{z};a,\frac{b}{s})$$

This equation shows that the scaling of a gamma distribution with parameters a, b leads to another gamma distribution with parameters a, $\frac{b}{s}$. Since the Lorenz curve does not depend on scaling, the Lorenz curve must be independent of the parameter b.

4.3 Description of the EPM

In a base year let there be (x_j^i, y_j^i) points of the Lorenz curve $\breve{L}^i$ of country i, i. e. $y_j^i = \breve{L}^i\,(x_j^i)$.

In the first step, an income PDF $f^i(z;p^i)$ for each country i is determined. For this purpose the parameters p^i are estimated by adapting the Lorenz curves $L^i(z;p^i)$ with a least square fit:

$\min_{p^i} \left\{\sum_j \left(L^i\left(x_j^i;p^i\right) - y_j^i\right)^2\right\}$.

In the second step, for each country i an emission PDF $\tilde{f}^i$ is obtained by scaling the income PDF f^i.

$$\tilde{f}^i\left(\tilde{z};p^i\right) = s^i * f^i(s^i * \tilde{z};p^i)$$

with the scaling factor $s^i := \frac{\text{average emissions in country } i}{\text{average income in country } i}$ of country i.

In the third step, in each year t a cap CA_t is determined in such a way that the emissions in all countries yield the underlying global emissions in the year t (E_t):

$$\sum_i E_t^i = \sum_i P_t^i \left(\int_{-\infty}^{CA_t} z\, \tilde{f}^i(z;p^i)\, dz + CA_t \int_{CA_t}^{\infty} \tilde{f}^i(z;p^i)\, dz\right) = E_t.$$

Usually, it is assumed that each person earns a non-negative income. That is why the scaling in the second step is possible. However, when global emissions are negative a different transformation, which converts an income PDF, which is zero for negative incomes, into an emission PDF that addresses negative emissions, must be found. Such transformations are conceivable, but they are not indisputable.

5. List of abbreviations

B_t	global emissions until the year t (global budget until the year t)
B_t^i	emissions of country i until the year t (national budget of country i until the year t)
BY	base year (space of time)
$\widehat{C}_t$	weighting of population in the year t in C&C
$\widetilde{C}_t$	weighting of population in the year t in LIMITS
$\overline{C}_t$	weighting of population in the year t in the RF
$\check{C}_t^i$	weighting of population of country i in the year t used to obtain the nation budget of country i
C&C	Contraction and Convergence Model
CA_t	cap in the year t
CDC	Common but Differentiated Convergence Model
CY	convergence year
E_{BY}	global emissions in the base year
E_{BY}^i	emissions of country i in the base year
E_{CY}	global emissions in the convergence year
E_{CY}^i	emissions of country i in the convergence year
E_t	global emissions in the year t
E_t^i	emissions of country i in the year t
$\widehat{E_t^i}$	emissions of country i in the year t in C&C
$\widetilde{E_t^i}$	emissions of country i in the year t in LIMITS
$\overline{E_t^i}$	emissions of country i in the year t in the RF
$E_t^{i\_bau}$	emissions of country i in the year t in a business-as-usual scenario
E_t^{oTH}	remaining global emissions in the year t for the countries over the threshold in the year t
$E_{t-1}^{oTH\_t}$	emissions in the year t – 1 of the countries over the threshold in the year t

$\dot{E}^i(z)$	emission power emission power (the derivative of emissions with respect to time, emissions per unit of time) of country *i* at a point of time *z*
$\dot{E}^i_{BY+1}$	emission power of country *i* at the end of the base year
EPM	Emission Probability Model
ESPM	Extended Smooth Pathway Model
f^i	income PDF of country *i*
$\tilde{f}^i$	emission PDF of country *i*, scaled PDF
F	cumulative distribution function, i. e. the probability of an income equal to z or less is $F(z) = \int_{-\infty}^{z} f(t)\, dt$
F^{-1}	inverse function of the cumulative distribution function *F*
$f^i(z;p^i)$	assumed income PDF of country *i* with parameters p^i to be estimated
$\tilde{f}^i(z;p^i)$	estimated emission PDF of country *i* with parameters p^i
G-C&C	Generalised C&C
G-Limits	Generalised LIMITS
GSPF	General Smooth Pathway Formula
i	country
m^i	mitigation rate (or the decay parameter) of country *i*
L	explicit representation of the Lorenz curve
$\bar{L}$	parametric representation of the Lorenz curve
$\check{L}^i$	Lorenz curve of country *i*
LIMITS	LIMITS Model
P_{CY}	global population in the convergence year
P^i_{CY}	population of country *i* in the convergence year
P_t	global population in the year *t*
P^i_t	population of country *i* in the year *t*
P^{oTH}_t	population in the year *t* of the countries over the threshold in the year *t*

PDF	probability density function
PT	percentage
r^i	change rate of the emission power of country i at the end of the base year $\left(\frac{d\dot{E}^i}{dz}(BY+1)/\dot{E}^i(BY+1) = r^i\right)$
RB	global remaining budget
RB^i	remaining budget of country *i*
RF	Regensburg Formula
s	scaling factor
s^i	scaling factor of country i $\left(\frac{\text{average emissions in country } i}{\text{average income in country } i}\right)$
SPF	Smooth Pathway Formulae
SPFR	Smooth Pathway Formula from Raupach et al.
t	year
T^i	time defined by the remaining budget of country *i* and the emission power of country *i* at the end of the base year $\left(T^i = \frac{RB^i}{\dot{E}^i_{BY+1}}\right)$
TH_t	threshold in the year *t*
(x^i_j, y^i_j)	points of the Lorenz curve $\check{L}^i$ of country i, i. e. $y^i_j = \check{L}^i(x^i_j)$
z	point of time (SPF), income (EPM)

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