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Mathematical Details on

Resource Sharing Models

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1 Introduction

According to the Paris Agreement, countries must regularly present their ambitions, i. e. their plans of CO_2 emissions (NDCs). This leads quickly to the question, on which criteria their ambitions are based.

Several models have been developed so far. Effort or burden sharing models can be understood as an answer to the question which is the effort of each country so that the business as usual pathway of each country is changed in a way to obtain in sum a pathway that meets the remaining global budget.[1]

We should like to concentrate on resource sharing models who give a direct answer to the question how the global remaining budget is shared. In some models, the point of time can be chosen when global emissions are attributed to countries according to population. In other models this point is at infinity when global and all national emissions are zero.

As a consequence, resource sharing models attribute all countries positive/negative emissions when the global emissions are positive/negative. On the other side effort sharing models can attribute in the same year some countries positive and some countries negative emissions.

Our aim is to facilitate the comparison of the resource sharing models putting the spotlight on the mathematical formulae and on an Excel tool (download: http://downloads.save-the-climate.info/ or www.klima-retten.info/downloads.html) comparing the resource sharing models for three typical countries.

[1] An elaboration of the Australian Climate Change Authority provides an initial overview: www.climatechangeauthority.gov.au/appendix-c-sharing-global-emissions-budget.

2 Models with convergence at a certain point of time

All models with convergence at a certain point of time start with a global pathway that meets a remaining global budget corresponding to a certain global warming. The idea is to start in each country in a base year (*BY*) with the actual emissions and to transform these emissions into emissions based on a per capita attribution during a convergence period ending with a convergence year (*CY*).

2.1 Per Capita Model

This is the easiest model. In the years before the convergence year each country is attributed emissions according to the emissions in the base year. In the convergence year each country is attributed emissions according to population. This leads to a rough transition in the convergence year.

$$E_t^i := \begin{cases} \frac{E_{BY}^i}{E_{BY}} * E_t, \text{ for } BY + 1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t, \text{ for } CY \leq t \end{cases} \tag{1}$$

2.2 Contraction & Convergence Model[2]

The Contraction & Convergence Model, the LIMITS Model and the Regensburg Formula are similar. Each model continuously replaces the attribution "ratio in the past" with the attribution "population" within a convergence period. As of the convergence year, only the attribution "population" is applied. However, the underlying formulae are different in each model.

The Contraction & Convergence formula

$$E_t^i := \begin{cases} \left((1 - C_t) * \frac{E_{t-1}^i}{E_{t-1}} + C_t * \frac{P_t^i}{P_t}\right) * E_t, \text{ for } BY + 1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t, \text{ for } CY \leq t \end{cases} \tag{2}$$

was already propounded by the Global Commons Institute in the early 1990s. C_t is any non-decreasing weighting function that takes the value 0 in the base year (*BY*) and the value 1 in the convergence year (*CY*). The most popular variants for C_t are

- exponential: $C_t = exp\left(-a\left(1 - \frac{t-BY}{CY-BY}\right)\right)$ with the parameter $a > 0$ to be determined. Note that $C_{BY} = exp(-a)$ is only approximately zero. For $a > 4$ population has the least influence compared with the other variants.
- konvex quadratic: $C_t = \left(\frac{t - BY}{CY-BY}\right)^2$; population has more influence than in the exponential variant with $a > 4$.
- linear: $C_t = \frac{t - BY}{CY-BY}$; population has more influence than in konvex quadratic variant.
- konkav quadratic: $C_t = 1 - \left(1 - \frac{t - BY}{CY-BY}\right)^2$; populations has more influence than in the linear variant.

[2] (cf.Meyer, n.d.)

The Global Commons Institute considered only the linear und the exponential form for C_t.

2.3 LIMITS Model

LIMITS, a research project funded by the EU, uses the following formula for the emissions of the country *i* in the year *t*:[3]

$$E_t^i := \begin{cases} \left((1-C_t) * \frac{E_{BY}^i}{E_{BY}} + C_t * \frac{P_t^i}{P_t}\right) * E_t, \text{for } BY + 1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t, \text{ for } CY \leq t \end{cases} \tag{3}$$

where C_t is a non decreasing weighting function that takes the value 0 in the base year and the value 1 in the convergence year. Although all of the variants for C_t mentioned in 2.2 are conceivable, LIMITS only considered the linear form.

2.4 Common but differentiated convergence Model (CDC)[4]

The “common but differentiated convergence” aproach refines the Contraction & Convergence Model. “This approach [CDC] eliminates two concerns often voiced in relation to gradually converging per-capita emissions: (i) advanced developing countries have their commitment to reduce emissions delayed […] (ii) CDC does not provide excess emission allowances to the least developing countries.” (Höhne, et al., 2006, p. 181) This is achieved by attributing contries below a continuously decreasing threshold emissions according to their free desicion noted in a business as usual (bau) scenario. Thus the Contraction & Convergence Model is only applied for countries with per capita emissions above this threshold.

In detail: First we define a threshold TH_t in the year *t* that decreases if the global emissions decrease:

$$TH_t := \frac{E_t}{P_t} * PT,$$

where PT is a given percentage, e. g. 0.95. If the average emissions of country *i* in the year *t* in a business as usual scenario are below or equal to the threshold, i. e. $\frac{E_t^{i\_bau}}{P_t^i} \leq TH_t$, the country is attributed emissions according to the business as usual scenario and we set

$$E_t^i := E_t^{i\_bau}.$$

Otherwise, i. e. if the average emissions of country *i* in the year *t* in the business as usual scenario are above the threshold ($\frac{E_t^{i\_bau}}{P_t^i} > TH_t$), a country is attributed emissions according to the contraction and convergence formula and we set

$$E_t^i := \left((1-C_t) * \frac{E_{t-1}^i}{EP_{t-1}^{oTH}} + C_t * \frac{P_t^i}{P_t^{oTH}}\right) * E_t^{oTH},$$

where

C_t is a non-decreasing weighting function,

[3] LIMITS uses the formula to determine emissions pathways for different regions of the world (cf. Tavoni, et al., 2013).

[4] (Cf. Höhne, et al., 2006). Unfortunately this source doesn’t contain any formulae. So the formulae we present are our interpretation of the describtion of the CDC Model.

E_t^{oTH} are the remaining emissions in the year *t*
for the countries over the threshold in the year *t*, i. e.

$$E_t^{oTH} = E_t - \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} \leq TH_t}} E_t^i$$

EP_{t-1}^{oTH} are the emissions in the year *t*-1
of the countries over the threshold in the year *t*, i. e.

$$EP_{t-1}^{oTH} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} E_{t-1}^i.$$

P_t^{oTH} is the population in the year *t*
of the countries over the threshold in the year *t*, i. e.

$$P_t^{oTH} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} P_t^i,$$

Remark: Obviously the equation

$$E_t^{oTH} = \sum_{\substack{i \\ \text{if } \frac{E_t^{i\_bau}}{P_t^i} > TH_t}} E_t^i,$$

holds, but this equation can't be used to define E_t^{oTH}, because E_t^i is defined with the help of E_t^{oTH}.

2.5 Regensburg Model[5]

In the Regensburg Formula the emissions of the country *i* in the year *t* are given by

$$E_t^i := \begin{cases} (1 - C_t) * E_{BY}^i + C_t * E_{CY}^i, \text{ for } BY + 1 \leq t < CY \\ \frac{P_t^i}{P_t} * E_t, \text{ for } CY \leq t \end{cases} \tag{4}$$

where $C_t = \dfrac{E_{BY} - E_t}{E_{BY} - E_{CY}}$ and $E_{CY}^i = \dfrac{E_{CY}}{P_{CY}} * P_{CY}^i$.

This formula should only be used when global emissions are decreasing to ensure increasing C_t. For other representations of this formula or for a proof that $\sum_i E_t^i = E_t$ (cf. Wittmann & Wolfsteiner, current version).

[5] (Cf. Sargl, et al., 2016).

The Regensburg Model similar to the CDC Model does not provide excess emission allowances to the least developing countries (cf. chapter 2.4). Since the Regensburg Model is the most favourabel convergence model for industrial countries we know, national emission pathways calculated with the Regensburg Formula describe a kind of floor of ambition for these countries. Industrial countries are in difficulty to explain their NDC if it falls below this floor.

The Regensburg Model can also be combined with the idea of the CDC Modell, that some countries are exempt from the emission attribution regime as long as their emissions are below a threshold. In this case “global” in the describtion of the formula above must be read as “of the countires above the threshold”.

3 Models with convergence at infinity

3.1 Smooth Pathway Model

Raupach et al.[6] showed how to transform an allocated remaining budget of the country *i* (RB^i) into a pathway with a smooth transition from the current pathway and with near-zero emissions in the future. Thus,

- in contrast to the other resource sharing models, the global pathway is obtained by summing up the pathways of all countries.
- the Smooth Pathway Model can't account for planned negative emissions of some countries in the future.

This model gives no hint how to obtain a remaining budget for each country. Of course, the global remaining budget can be attributed to countries according to population or emissions in the past.

In the Smooth Pathway Model for the emission of the country *i* at the time *z* the following function is used

$$E^i(z) \;=\; E^i_{BY}\big(1 + \;(r^i + m^i)(z - BY)\big)e^{-m^i(z-BY)}, \tag{5}$$

where

E^i_{BJ} are the emissions of country *i* at the beginning of the base year ($E^i(BY) \;=\; E^i_{BY}$),

r^i is the rate of change of emissions of country *i* at the beginning of the base year ($\frac{E^{i\prime}(BY)}{E^i(BY)} = r^i$)

m^i is the mitigation rate (or the decay parameter) of country *i*.

The mitigation rate m^i is determined such that the allocated remaining budget of country *i* (RB^i) is met:

$$\int_{BY}^{\infty} E^i(z)\,dz \;=\; RB^i$$

Thus, if $r^i > -1/T^i$, the mitigation rate m^i is given by

$$m^i \;=\; \frac{1 + \sqrt{1 + r^i T^i}}{T^i},$$

where $T^i \;=\; \frac{RB^i}{E^i_{BY}}$ is the emission time defined by the remaining budget of country *i* and the emissions of country *i* at the beginning of the base year. Otherwise, there is no solution for the mitigation rate m^i.

Since we are more interested in the emissions of country *i* in the year *t* (E^i_t) than in the emissions at a point of time *z*, we integrate equation (5) and obtain:

[6] (Cf. Raupach, et al., 2014). Free supplementary information containing mathematical details on the properties of the formula in equation (5) can be retrieved from http://www.nature.com/nclimate/journal/v4/n10/extref/nclimate2384-s1.pdf.

$$E_t^i = \int_{t-1}^{t} E^i(z)\,dz = -E_{BY}^i \frac{e^{-m^i(t-BY)}}{(m^i)^2}\left[\left(r^i m^i + \left(m^i\right)^2\right)(t-BY) + 2m^i + r^i\right]$$
$$+E_{BY}^i \frac{e^{-m^i(t-BY-1)}}{(m^i)^2}\left[\left(r^i m^i + \left(m^i\right)^2\right)(t-BY-1) + 2m^i + r^i\right].$$

3.2 Emission Probability Model

From a mathematical point of view, this model is challenging because it does not only take emissions and populations into consideration but also income probabilities.

3.2.1 The basic steps

Chakravarty et al.[7] described three steps how to obtain and cut an emission probability density function starting with the points of a Lorenz curve.

Let $\left(x_j^i, y_j^i\right)$ be points of the Lorenz curve $\check{L}^i$ of country i, i. e. $y_j^i = \check{L}^i\,(x_j^i)$, and $\check{Z}^i$ be random variables representing the income of a person in the country *i*.

In a first step, the parameters p^i of an assumed income probability density function (PDF) $f^i(z; p^i)$ for each country *i* is estimated by fitting the Lorenz curves $L^i(z; p^i)$ with a least square fit:

$$\min_{p^i}\left\{\sum_j \left(L^i(x_j^i; p^i) - y_j^i\right)^2\right\}.$$

In a second step income random variables Z^i are scaled to obtain emission random variables $\tilde{Z}^i$:

$$\tilde{Z}^i = s^i * Z^i$$

with the scaling factor $s^i \coloneqq \frac{\text{average emissions in country } i}{\text{average income in country } i}$ of country i.

In a third step in each year *t* a cap CA_t is determined such that the emission in all countries yield the agreed upon global emissions in the year *t* (E_t):

$$\sum_i P_t^i \left(\int_{-\infty}^{CA_t} z\,\tilde{f}^i(z; p^i)\,dz + CA_t \int_{CA_t}^{\infty} \tilde{f}^i(z; p^i)\,dz\right) = E_t.$$

The emissions of the country *i* in the year t are then given by

$$E_t^i = P_t^i \left(\int_{-\infty}^{CA_t} z\,\tilde{f}^i(z; p^i)\,dz + CA_t \int_{CA_t}^{\infty} \tilde{f}^i(z; p^i)\,dz\right).$$

Global negative emissions: Normally, it is assumed that each person earns a positive income. That is why the scaling in the second step is possible. However, when global emissions are negative an other transformation must be found that converts an income PDF, which is zero for nega-

[7] (Cf. Chakravartya, et al., 2009).

tive incomes, into an emission PDF that addresses negative emissions. Such transformations are conceivable, but they are not indisputable.

3.2.2 Excursion: Probability density function and Lorenz Curve

For some readers a summary on how to obtain a Lorenz Curve from a probability function might be useful.

Let *f* be an income probability density function.

Then

- *f* takes for the income *z* the probability value $f(z)$,
- the cumulative income share *x* is given by the cumulative distribution function (CDF) *F*, i. e. the probability for an income equal to z or less is $x = F(z) = \int_{-\infty}^{z} f(t)\,dt$ and
- a parametric representation of the Lorenz curve $\bar{L}$ is given by

$$\bar{L}(z) = \begin{pmatrix} x = F(z) \\ y = \dfrac{\int_{-\infty}^{z} t\,f(t)\,dt}{\int_{-\infty}^{\infty} t\,f(t)\,dt} \end{pmatrix} \qquad (6)$$

$\int_{-\infty}^{z} t\,f(t)\,dt$: average income of the persons with an income equal to *z* or less

$\int_{-\infty}^{\infty} t\,f(t)\,dt$: average income of the population

For a CDF *F* with the inverse function F^{-1} the Lorenz curve *L* is directly given by

$$y = L(x) = \frac{\int_{-\infty}^{F^{-1}(x)} t\,f(t)\,dt}{\int_{-\infty}^{\infty} t\,f(t)\,dt}. \qquad (7)$$

Substituting $t = F^{-1}(\check{t})$ yields $\frac{dt}{d\check{t}} = (F^{-1})'(\check{t}) = \frac{1}{F'(F^{-1}(\check{t}))} = \frac{1}{f(F^{-1}(\check{t}))}$ and the Lorenz curve can be written as

$$y = L(x) = \frac{\int_{0}^{x} F^{-1}(\check{t})d\check{t}}{\int_{0}^{1} F^{-1}(\check{t})d\check{t}}. \qquad (8)$$

Scaling Theorem: The Lorenz curve is independent of the scaling of the z-axis.

Proof: With a scaling factor $s \neq 0$ the scaled PDF $\tilde{f}$ for a PDF *f* is given by

$$\tilde{f}(\tilde{z}) = s\,f(s\tilde{z}).$$

For the CDF $\tilde{F}$ we obtain

$$\tilde{F}(\tilde{z}) = \int_{-\infty}^{\tilde{z}} \tilde{f}(\tilde{t})\,d\tilde{t} = s\int_{-\infty}^{\tilde{z}} f(s\tilde{t})\,d\tilde{t} = \int_{-\infty}^{s\tilde{z}} f(t)dt = F(s\tilde{z}).$$

Thus $\tilde{F}^{-1}$, the inverse function of the CDF $\tilde{F}$, is given by

$$\tilde{F}^{-1} = \frac{1}{s}\,F^{-1}.$$

With the help of the representation (8) of the Lorenz curve we see that, the Lorenz curve from the PDF *f* and the PDF $\tilde{f}$ are the same.

3.2.3 Special case: gamma probability distribution

In general, the evaluation of the integrals in equation (7) or (8) can cause trouble. However if Z is a gamma distributed random variable all this work can be done by a spreadsheet programme, such as EXCEL.

Let Z be a gamma distributed random variable. Then the PDF g is given by

$$g(z; a, b) = \begin{cases} 0 & \text{for } z < 0 \\ \frac{1}{b^a \Gamma(a)} z^{a-1} e^{-\frac{z}{b}} & \text{for } z \geq 0 \end{cases}$$

with parameters a, $b > 0$ and $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z}\, dz$.

The CDF is denoted by

$$G(z; a, b) = \int_0^z g(t;\ a, b)\, dt = \int_0^z \frac{1}{b^a \Gamma(a)} t^{a-1} e^{-\frac{t}{b}}\, dt$$

Since $\Gamma(a+1) = a\,\Gamma(a)$, the equation $t\, g(t; a, b) = ab\, g(t; a+1, b)$ holds. Thus

- the expected value (or mean) of Z is given by

$$E[Z] = \int_0^\infty t\ g(t; a, b)\, dt = ab \int_0^\infty g(t; a+1, b) dt = ab$$

  and

- using the representation (7) the Lorenz curve is given by

$$L(x) = \frac{\int_0^{G^{-1}(x;a,b)} t\, g(t; a, b)\, dt}{\int_0^\infty t\, g(t, a, b)\, dt} = \frac{ab\ \int_0^{G^{-1}(x;a,b)} g(t; a+1, b)\, dt}{ab}$$
$$= G(G^{-1}(x; a, b); a+1, b).$$

<u>Scaling</u>

With a scaling factor $s \neq 0$ we easily find

$$\tilde{g}(\tilde{z}; a, b) = s\ g(s\tilde{z}; a, b) = g(\tilde{z}; a, \frac{b}{s})$$

This equation shows that the scaling of a gamma distribution with parameters a, b leads to another gamma distribution with parameters a, $\frac{b}{s}$. Since the Lorenz curve does not depend on scaling, the Lorenz curve must be independent of the parameter b.

4 List of abbreviations

BY	base year
C_t	weighting of per capita emissions in the year t
CA_t	cap in the year t
CY	convergence year
E_{BY}	global emissions in the base year
E_{BY}^i	emissions of country *i* in the base year
E_{BY}^i	emissions of country *i* at the beginning of the base year ($E^i(BY) = E_{BY}^i$)
E_{CY}	global emissions in the convergence year
E_{CY}^i	emissions of country *i* in the convergence year
E_t	global emissions in the year *t*
E_t^i	emissions of country *i* in the year *t*
$E_t^{i\_bau}$	emissions of country *i* in the year *t* in a business as usual scenario
E_t^{oTH}	remaining global emissions in the year *t* for the countries over the threshold in the year *t*
EP_{t-1}^{oTH}	global emissions in the year *t* – 1 for the countries over the threshold in the year *t*
$E^i(z)$	emissions of country *i* at the time *z*
$E[Z]$	expected value (of mean) of the random variable Z
f	income probability density function
$\tilde{f}$	scaled pdf, emission probability density function
F	cumulative distribution function, i. e. the probability for an income equal to z or less is $F(z) = \int_{-\infty}^{z} f(t)\, dt$
$\tilde{F}$	cumulative distribution function, the corresponding PDF is $\tilde{f}$
F^{-1}	inverse function of the cumulative distribution function F
$\tilde{F}^{-1}$	inverse function of the cumulative distribution function $\tilde{F}$
$f^i(z; p^i)$	assumed income probability density function (PDF) of country *i* with parameters p^i to be estimated
$\tilde{f}^i(z; p^i)$	estimated emission probability density function (PDF) of country *i* with parameters p^i
$g(z; a, b)$	PDF of a gamma distributed random variable with parameters a, $b > 0$
$G(z; a, b)$	CDF of a gamma distributed random variable with parameters a, $b > 0$
m^i	mitigation rate (or the decay parameter) of country *i*
L	explicit representation of the Lorenz curve
$\bar{L}$	parametric representation of the Lorenz curve
P_{CY}	global population in the convergence year
P_{CY}^i	population of country *i* in the convergence year

P_t	global population in the year *t*
P_t^i	population of country *i* in the year *t*
P_t^{oTH}	population in the year *t* of the countries over the threshold in the year *t*
PT	percentage
r^i	rate of change of emissions of country *i* at the beginning of the base year ($\frac{E^{i\prime}(BY)}{E^i(BY)} = r^i$)
RB^i	remaining budget of country *i*
s	scaling factor
s^i	scaling factor of country i $\left(\frac{\text{average emissions in country } i}{\text{average income in country } i}\right)$
T^i	emission time defined by the remaining budget of country *i* and the emissions of country *i* at the beginning of the base year $\left(T^i = \frac{RB^i}{E_{BY}^i}\right)$
TH_t	threshold in the year *t*
(x_j^i, y_j^i)	points of the Lorenz curve $\check{L}^i$ of country i, i. e. $y_j^i = \check{L}^i\ (x_j^i)$
Z^i	estimated random variables representing the income of a person in the country *i*
$\check{Z}^i$	random variables representing the income of a person in the country *i*
$\tilde{Z}^i$	estimated random variables representing the emisssions of a person in the country *i*

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